2

Production With One Variable Input

This chapter introduces the concept of a production function and uses the concept as a basis for the development of the factor-product model. An agricultural production function in presented using graphical and tabular approaches. Algebraic examples of simple production functions with one input and one output are developed. Key features of the neoclassical production function are outlined. The concept of marginal and average physical product is introduced. The use of the first, second, and third derivatives in determining the shape of the underlying total, marginal, and average product is illustrated, and the concept of the elasticity of production is presented.

Key terms and definitions:

Production Function
Domain
Range
Continuous Production Function
Discrete Production Function
Fixed Input
Variable Input
Short Run
Long Run
Intermediate Run
Sunk Costs
Law of Diminishing (Marginal) Returns
Total Physical Product (*TPP*)
Marginal Physical Product (*MPP*)
Average Physical Product (*APP*)
$\Delta y/\Delta x$
Sign
Slope
Curvature
First Derivative
Second Derivative
Third Derivative
Elasticity of Production

2.1 What Is a Production Function?

A production function describes the technical relationship that transforms inputs (resources) into outputs (commodities). A mathematician defines a *function* as a rule for assigning to each value in one set of variables (the domain of the function) a single value in another set of variables (the range of the function).

A general way of writing a production function is

†2.1 $$y = f(x)$$

where y is an output and x is an input. All values of x greater than or equal to zero constitute the domain of this function. The range of the function consists of each output level (y) that results from each level of input (x) being used. Equation †2.1 is a very general form for a production function. All that is known about the function $f(x)$ so far is that it meets the mathematician's definition of a function. Given this general form, it is not possible to determine exactly how much output (y) would result from a given level of input (x). The specific form of the function $f(x)$ would be needed, and $f(x)$ could take on many specific forms.

Suppose the simple function

†2.2 $$y = 2x.$$

For each value of x, a unique and single value of y is assigned. For example if $x = 2$, then $y = 4$; if $x = 6$ then $y = 12$ and so on. The domain of the function is all possible values for x, and the range is the set of y values corresponding to each x. In equation †2.2 , each unit of input (x) produces 2 units of output (y).

Now consider the function

†23 $$y = \sqrt{x}$$

It is not possible to take the square root of a negative number and get a real number. Hence the domain (x) and range (y) of equation †2.3 includes only those numbers greater than or equal to zero. Here again the function meets the basic definition that a single value in the range be assigned to each value in the domain of the function. This restriction would be all right for a production function, since it is unlikely that a farmer would ever use a negative quantity of input. It is not clear what a negative quantity of an input might be.

Functions might be expressed in other ways. The following is an example:

If $x = 10$, then $y = 25$.
If $x = 20$, then $y = 50$.
If $x = 30$, then $y = 60$.
If $x = 40$, then $y = 65$.
If $x = 50$, then $y = 60$.

Notice again that a single value for y is assigned to each x. Notice also that there are two values for x (30 and 50) that get assigned the same value for y (60). The mathematician's definition of a function allows for this. But one value for y must be assigned to each x. It does not matter if two different x values are assigned the same y value.

The converse, however, is not true. Suppose that the example were modified only slightly:

If $x = 25$, then $y = 10$.
If $x = 50$, then $y = 20$.
If $x = 60$, then $y = 30$.
If $x = 65$, then $y = 40$.
If $x = 60$, then $y = 50$.

This is an example that violates the definition of a function. Notice that for the value $x = 60$, two values of y are assigned, 30 and 50. This cannot be. The definition of a function stated that a single value for y must be assigned to each x. The relationship described here represents what is known as a *correspondence*, but not a function. A correspondence describes the relationship between two variables. All functions are correspondences, but not all correspondences are functions.

Some of these ideas can be applied to hypothetical data describing the production of corn in response to the use of nitrogen fertilizer. Table 2.1 represents the relationship and provides specific values for the general production function $y = f(x)$. For each nitrogen application level, a single yield is defined. The yield level is sometimes referred to as the total physical product (*TPP*) resulting from the nitrogen that is applied.

Table 2.1 Corn Yield Response to Nitrogen Fertilizer

Quantity of Nitrogen (Pounds/Acre)	Yield in Bushels/Acre
0	50
40	75
80	105
120	115
160	123
200	128
240	124

From Table 2.1, 160 pounds of nitrogen per acre will result in a corn yield or *TPP* of 123 bushels per acre. The concept of a function has a good deal of impact on the basic assumptions underlying the economics of agricultural production.

Another possible problem exists with the interpretation of the data contained in Table 2.1. The exact amount of corn (*TPP*) that will be produced if a farmer decides to apply 120 pounds of nitrogen per acre can be determined from Table 2.1, but what happens if the farmer decides to apply 140 pounds of nitrogen per acre? A yield has not been assigned to this nitrogen application level. A mathematician might say that our production function $y = f(x)$ is discontinuous at any nitrogen application level other than those specifically listed in Table 2.1.

A simple solution might be to interpolate between the known values. If 120 pounds per acre produces 115 bushels of corn, and 160 pounds of nitrogen produces 123 bushels of corn, the yield at 140 pounds might be $(115 + 123)/2$ or 119 bushels per acre. However, incremental increases in nitrogen application do not provide equal incremental increases in corn production throughout the domain of the function. There is no doubt that some nitrogen is available in the soil from decaying organic material and nitrogen applied in previous seasons, and nitrogen need not be applied in order to get back the first 50 bushels of corn.

The first 40 pounds of nitrogen applied produces 25 additional bushels, for a total of 75 bushels, the next 40 pounds produces 30 bushels of corn, for a total of 105 bushels, but the productivity of the remaining 40 pound increments in terms of corn production declines. The next 40 pounds increases yield by only 10 bushels per acre, the 40 pounds after that by only 8 bushels per acre, and the final 40 pounds by only 5 bushels per acre.

Following this rationale, it seems unlikely that 140 pounds of nitrogen would produce a yield of 119 bushels, and a more likely guess might be 120 or 121 bushels. These are only guesses. In reality no information about the behavior of the function is available at nitrogen application levels other than those listed in Table 2.1. A yield of 160 bushels per acre at a nitrogen application level of 140 pounds per acre could result- or, for that matter, any other yield.

Suppose instead that the relationship between the amount of nitrogen that is applied and corn yield is described as

†2.4 $$y = 0.75x + 0.0042x^2 - 0.000023x^3$$

where y = corn yield (total physical product) in bushels per acre

x = nitrogen applied in pounds per acre

Equation †2.4 has some advantages over the tabular function presented in Table 2.1. The major advantage is that it is possible to calculate the resultant corn yield at any fertilizer application level. For example, the corn yield when 200 pounds of fertilizer is applied is $0.75(200) + 0.0042(200^2) - 0.000023(200^3) =$ 134 bushels per acre.

Moreover, a function such as this is continuous. There are no nitrogen levels where a corn yield cannot be calculated. The yield at a nitrogen application level of 186.5 pounds per acre can be calculated exactly. Such a function has other advantages, particularly if the additional output resulting from an extra pound of nitrogen is to be calculated. The yields of corn at the nitrogen application rates shown in Table 2.1 can be calculated and are presented in Table 2.2.

Table 2.2 Corn Yields at Alternative Nitrogen Application Rates for the Production Function $y = 0.75x + 0.0042x^2 - 0.000023x^3$

Quantity of Nitrogen, x (lb/acre)	Corn Yield, y or TPP (bu/Acre)
0	0.0
20	16.496
40	35.248
60	55.152
80	75.104
100	94.000
120	110.736
140	124.208
160	133.312
180	136.944
200	134.000
220	123.376
240	103.968

The corn yields (*TPP*) generated by the production function in Table 2.2 are not the same as those presented in Table 2.1. There is no reason for both functions to generate the same yields. A continuous function that would generate exactly the same yields as those presented in Table 2.1 would be very complicated algebraically. Economists like to work with continuous functions, rather than discrete production functions from tabular data, in that the yield for any level of input use can be readily obtained without any need for interpolation. However, a tabular presentation would probably make more sense to farmers.

The yields generated in Table 2.2 also differ from those in Table 2.1 in another important way. Table 2.1 states that if a farmer applied no nitrogen to corn, a yield of 50 bushels per acre is obtained. Of course, nitrogen is absolutely essential for corn to grow. As indicated earlier, the data contained in Table 2.1 assume that there is some residual nitrogen in the soil on which the corn is grown. The nitrogen is in the soil because of decaying organic material and leftover nitrogen from fertilizers applied in years past. As a result, the data in Table 2.1 reveal higher yields at low nitrogen application levels than do the data contained in Table 2.2.

The mathematical function used as the basis for Table 2.2 could be modified to take this residual nitrogen into account by adding a constant such as 50. The remaining coefficients of the function (the 0.75, the 0.0042, and the !0.000023) would also need to be altered as well. Otherwise, the production function would produce a possible but perhaps unrealistic corn yield of 50 + 136.944 = 186.944 bushels per acre when 180 pounds of fertilizer were applied. For many production processes in agriculture, no input produces no output. Consider the case of the production of beef using feed as an input. No feed would indeed produce no beef. In the case of crop production, some yield will normally result without chemical fertilizers.

A production function thus represents the relationship that exists between inputs and outputs. For each level of input use, the function assigns a unique output level. When a zero level of input is used, output might be zero, or, in some instances, output might be produced without the input.

2.2 Fixed Versus Variable Inputs and the Length of Run

So far, examples have included only one input or factor of production. The general form of the production function was

†2.5 $y = f(x)$

where y = an output

x = an input

Equation †2.5 is an ultrasimplistic production function for agricultural commodities. Such a function assumes that the production process can be accurately described by a function in which only one input or factor of production is used to produce an output. Few, if any, agricultural commodities are produced in this manner. Most agricultural commodities require several, if not a dozen or more, inputs. As an alternative, suppose a production function where there are several inputs and all but one are assumed to be held fixed at some constant level. The production function would thus become

†2.6 $y = f(x_1, {}^*x_2, x_3, x_4, x_5, x_6, x_7)$.

For example, y might be the yield of corn in bushels per acre, and x_1 might represent the amount of nitrogen fertilizer applied per acre. Variables $x_2, ..., x_7$ might represent each of the other inputs used in the production of corn, such as land, labor, and machinery.

Thus, in this example, the input x_1 is treated as the "variable" input, while the remaining inputs ($x_2, ..., x_7$) are assumed to be held constant at some fixed level. The "*" can be read as the word "given". As the use of x_1 is "varied" or increased, units of the variable input x_1 are added to units of the fixed inputs $x_2, ..., x_7$.

How can it be determined if an input should be treated as fixed or variable? A *variable input* is often thought of as an input that the farm manager can control or for which he or she can alter the level of use. This implies that the farmer has sufficient time to adjust the amount of input being used. Nitrogen in corn production has often been cited as an example of a variable input, in that the farmer can control the amount to be applied to the field.

A *fixed input* is usually defined as an input which for some reason the farmer has no control over the amount available. The amount of land a farmer has might be treated as a fixed input.

However, these distinctions become muddy and confused. Given sufficient time, a farmer might be able to find additional land to rent or purchase, or the farmer might sell some of the land owned. If the length of time were sufficient to do this, the land input might be treated as a variable input.

The categorization of inputs as either fixed or variable is closely intertwined with the concept of time. Economists sometimes define the *long run* as time of sufficient length such that all inputs to the production function can be treated as variable. The *very short run* can be defined as a period of time so short that none of the inputs are variable. Other lengths of time can also be defined. For example, the *short run* is a period of time long enough such that a few of the inputs can be treated as variable, but most are fixed. The *intermediate run* is long enough so that many, but not all inputs are treated as variable.

These categories again are somewhat arbitrary. If an economist were asked "How long is the short run?", the answer would probably be that the short run is a period of time sufficiently long that some inputs can be treated as variable, but sufficiently short such that some inputs can be treated as fixed. Does this imply a length of time of a day, a week, a month, or a crop production season? The length of time involved could be any of these.

Once fertilizer has been applied, a farmer no longer has control over application levels. The input that was previously classified as variable becomes fixed. Seed before planting is classified as a variable input. Once it is planted in the ground, seed can no longer be treated as a variable input.

Some production economists have argued that inputs should not be arbitrarily categorized as either fixed or variable. These arbitrary categories can be highly misleading. Production economists argue that in the case of crop production, prior to planting, nearly all inputs are variable. Farmers might rent additional land, buy or sell machinery, or adjust acreages of crops. Here is where real decision making can take place. Once planting begins, more and more of the inputs previously treated as variable become fixed. Tractor time and labor for tillage operations cannot be recovered once used. Acreages of crops once planted largely cannot be altered. Insecticides and herbicides are variable inputs before application, but must be treated as fixed or "sunk" once they have been applied. At the start of harvest, the only variable input is the labor, fuel, and repairs to run the harvesting equipment and to move the grain to market.

This view treats the input categories as a continuum rather than as a dichotomy. As inputs are used, costs are treated as sunk. Inputs, once used, can no longer be sold, or used on the farm for a different enterprise, such as another crop.

2.3 The Law of Diminishing Returns

The law of diminishing returns is fundamental to all of production economics. The law is misnamed. It should be called the law of diminishing *MARGINAL* returns, for the law deals with what happens to the incremental or marginal product as units of input or resource are added. The *law of diminishing marginal returns* states that as units of an variable input are added to units of one or more fixed inputs, after a point, each incremental unit of the variable input produces less and less additional output. As units of the variable input are added to units of the fixed inputs, the proportions change between fixed and variable inputs. The law of diminishing returns has sometimes been referred to as the law of variable proportions.

For example, if incremental units of nitrogen fertilizer were applied to corn, after a point, each incremental unit of nitrogen fertilizer would produce less and less additional corn. Were it not for the law of diminishing returns, a single farmer could produce all the corn required in the world, merely by acquiring all of the available nitrogen fertilizer and applying it to his or her farm.

The key word in the law of diminishing returns is *additional*. The law of diminishing returns does not state that as units of a variable input are added, each incremental unit of input produces less output in total. If it did, a production function would need to have a negative slope in order for the law of diminishing returns to hold. Rather, the law of diminishing returns refers to the rate of change in the slope of the production function. This is sometimes referred to as the *curvature* of the production function.

Figure 2.1 illustrates three production functions. The production function labeled A has no curvature at all. The law of diminishing returns does not hold here. Each incremental unit of input use produces the exact same incremental output, regardless of where one is at on the function. An example of a function such as this is

†2.7 $y = 2x$.

Each incremental unit of x produces 2 units of y, regardless of the initial value for x, whether it be 0, 24, 100 or 5000.

A slightly more general form of this function is

†2.8 $y = bx$.

where b is some positive number. If b is a positive number, the function is said to exhibit *constant marginal returns* to the variable input x, and the law of diminishing returns does not hold. Each incremental unit of x produces bx units of y.

The production function labeled B represents another kind of relationship. Here each incremental unit of x produces more and more additional y. Hence the law of diminishing returns does not hold here either. Notice that as the use of input x is increased, x becomes more productive, producing more and more additional y. An example of a function that would represent this kind of a relationship is

†2.9 $y = x^2$.

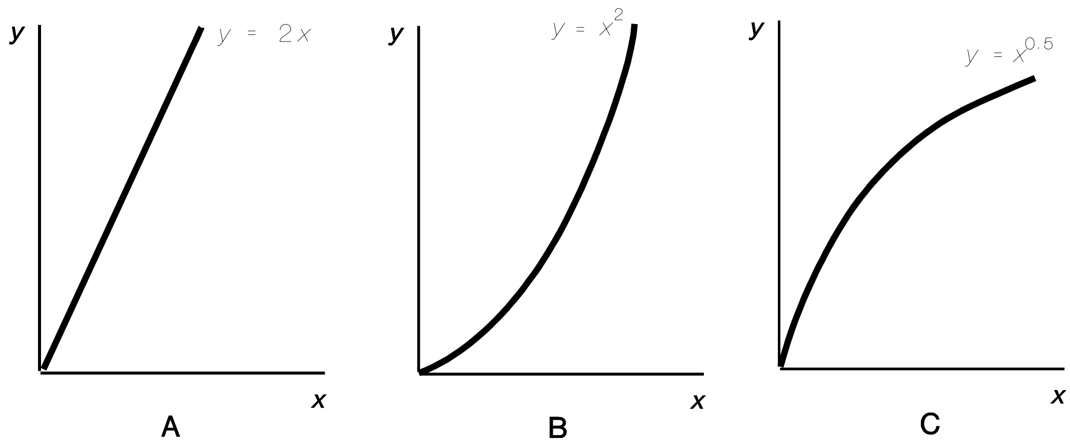

Figure 2.1 Three Production Functions

A slightly more general form of the function might be

†2.10 $y = ax^b$,

where both a and b are positive numbers, and b is greater than 1. Notice that if $b = 1$, the function is the same as the one depicted in diagram A of figure 2.1. The value of a must be positive if the input is to produce a positive quantity of output.

The production function labeled C represents the law of diminishing returns throughout its range. Here each incremental unit of x produces less and less additional y. Thus each unit of x becomes less and less productive. An example of a function that represents this kind of relationship is

†2.11 $y = \sqrt{x}$.

Another way of writing equation †2.11 is

†2.12 $y = x^{0.5}$.

Both are exactly the same thing. For this production function, total product (*TPP* or y) will never decline.

A slightly more general form of the function is

†2.13 $y = ax^b$,

where a and b are positive numbers. However, here b must be less than 1 but greater than zero, if diminishing (marginal) returns are to hold. This function will forever increase, but at a decreasing rate.

2.4 Marginal and Average Physical Product

The *marginal physical product* (*MPP*) refers to the change in output associated with an incremental change in the use of an input. The incremental increase in input use is usually taken to be 1 unit. Thus *MPP* is the change in output associated with a 1 unit increase in the input. The *MPP* of input x_i might be referred to as MPP_{x_i}. Notice that *MPP*, representing the incremental change in *TPP*, can be either positive or negative.

Average physical product (*APP*) is defined as the ratio of output to input. That is, $APP = y/x$. For any level of input use (x), *APP* represents the average amount of output per unit of x being used.

Suppose that the production function is

†2.14 $$y = f(x).$$

One way of expressing *MPP* is by the expression $\Delta y/\Delta x$, where the Δ denotes change. The expression $\Delta y/\Delta x$ can be read as "the change in y (Δy) with respect to a change in x (Δx)." For the same function *APP* is expressed either as y/x or as $f(x)/x$.

For the production function

†2.15 $$y = 2x,$$

MPP is equal to 2. The change in y with respect to a 1 unit change in x is 2 units. That is, each additional or incremental unit of x produces 2 additional or incremental units of y. For each additional unit of x that is used, *TPP* increases by 2 units. In this example *APP* equals y/x, or *APP* equals $2x/x$, or *APP* equals 2. For this simple production function $MPP = APP = 2$ for all positive values for x.

For the production function

†2.16 $$y = bx,$$

MPP is equal to the constant coefficient b. The change in y with respect to a change in x is b. Each incremental or additional unit of x produces b incremental or additional units of y. That is, the change in *TPP* resulting from a 1 unit change in x is b. Moreover, $APP = bx/x$. Thus, $MPP = APP = b$ everywhere.

Marginal and average physical products for the tabular data presented in Table 2.1 may be calculated based on the definition that *MPP* is the change in output (Δy) arising from an incremental change in the use of the input (Δx) and that *APP* is simply output (y) divided by input (x). These data are presented in Table 2.3. *MPP* is calculated by first making up a column representing the rate of change in corn yield. This rate of change might be referred to as Δy or perhaps ΔTPP. Then the rate of change in nitrogen use is calculated. This might be referred to as Δx. Since 40 pound units were used in this example, the rate of change in each case for x is 40. The corresponding *MPP* over the increment is $\Delta y/\Delta x$. *MPP* might also be thought of as $\Delta TPP/\Delta x$. The corresponding calculations are shown under the column labeled *MPP* in Table 2.3. For example, if nitrogen use increases from 120 to 160 pounds per acre, or 40 pounds, the corresponding increase in corn yield will be from 123 to 128 bushels per acre, or 5 bushels. The *MPP* over this range is approximately 5/40 or 0.125.

The *MPP*'s are positioned at the midpoint between each fertilizer increment. The *MPP*'s calculated here are averages that apply only approximately at the midpoints between each increment, that is at nitrogen application levels of approximately 20, 60, 100, 140 and 180 pounds per acre. Since no information is available with respect to what corn might have yielded at these midpoints, the calculated *MPP*'s are at best approximations that might in certain instances not be very accurate.

Table 2.3 also includes calculations for average physical product. Average physical product (*APP*) is defined as the ratio of output to input. That is, *APP* = *y*/*x*. For any level of input use (*x*), *APP* represents the average amount of output per unit of *x* being used. In Table 2.3, *APP* is calculated by dividing corn yield by the amount of nitrogen. These calculations are presented in the column labeled *APP*. The values for *APP* are exact at the specified levels of input use. For example, the exact *APP* when 120 pounds of nitrogen is applied is 115/120 or 0.958.

Table 2.3 *MPP* and *APP* for Corn Yield Response to Nitrogen Fertilizer

Quantity of Nitrogen (lb/acre)	)*x*	Yield of Corn (bu/acre)	)*y*	*MPP*	*APP*
0		50			50/0 = undefined
	40		25	25/40 = 0.625	
40		75			75/40 = 1.875
	40		30	30/40 = 0.75	
80		105			105/80 = 1.313
	40		10	10/40 = 0.25	
120		115			115/120 = 0.958
	40		8	8/40 = 0.20	
160		123			123/160 = 0.769
	40		5	5/40 = 0.125	
200		128			128/200 = 0.640
	40		!4	!4/40 = !0.100	
240		124			124/240 = 0.517

2.5 *MPP* and the Marginal Product Function

The procedure described in section 2.4 for calculating *MPP*'s is tedious and time consuming. There exists a quicker and more accurate means for calculating *MPP* and *APP* if the production function is given.

The *MPP* ( )*y*/)*x*) represents the slope or rate of change in the production function. The production function itself is sometimes referred to as total physical product (or *TPP*) function. The *MPP* function refers to the function representing the rate of change in the *TPP* function. If the slope of the *TPP* function were to be graphed, the result would be the *MPP* function, representing the rate of change in the *TPP* or the underlying production function as the use of variable input *x* is varied.

Given the *TPP* function (or production function), the *MPP* function (or marginal product function) might easily be obtained. Suppose again that the *TPP* or production function is represented by

†2.17 $y = 2x$

Again, the incremental increase in *y* associated with a 1 unit increase in the use of *x* is 2 units. Hence *MPP* = 2. Moreover,)*y*/)*x* = 2. In this case the marginal product function is equal to the constant 2.

For functions that do not have a constant slope, the expression $\Delta y/\Delta x$ can only approximate the slope of the function at a given point (Figure 2.2). The approximation can be very crude and inaccurate if a large value for Δx is chosen for the incremental change in *x*. This approximation improves as the value for Δx is chosen to be smaller and smaller. If the exact slope or *MPP* of a production function is to be found at a specific point, the magnitude of Δx must become infinitely small. That is, Δx must approach zero.

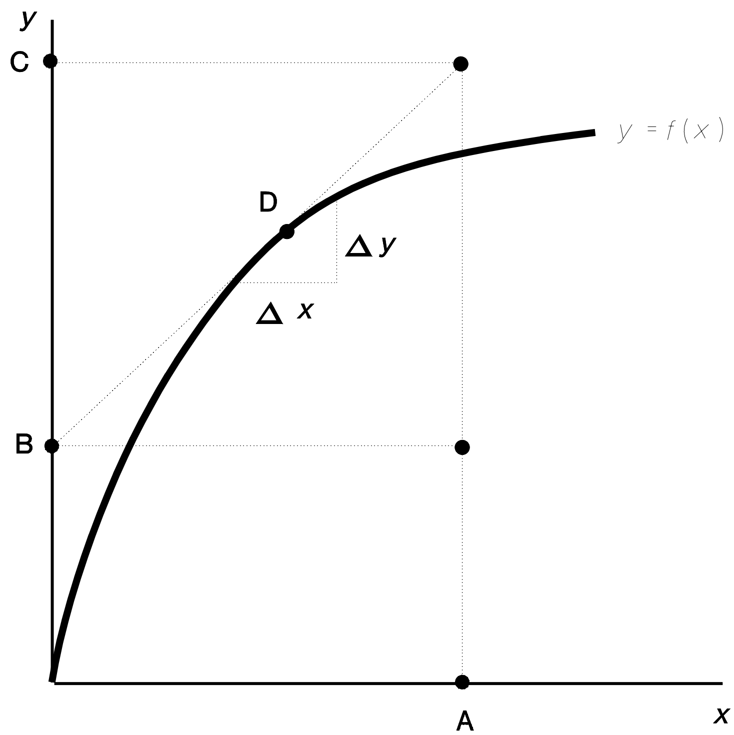

Figure 2.2 Approximate and Exact *MPP*

One way for finding the exact slope of a production function at a particular point is shown in Figure 2.2. Suppose that the exact *MPP* at point D is desired. A line is drawn tangent to the production function at D. which intersects the vertical axis at point B. The exact *MPP* at point D is equal to the slope of this line. This slope can be expressed as BC/OA. The graphical approach is time consuming, particularly if the *MPP* at several points along the function are to be calculated. A better way might be to find the first derivative of the production function. The *first derivative* of the production function is defined as the limit of the expression $\Delta y/\Delta x$ as Δx approaches zero. As Δx becomes smaller and smaller, $\Delta y/\Delta x$ becomes a better and better approximation of the true slope of the function. The first derivative, dy/dx, represents the exact slope of the production function at a particular point. In Figure 2.2, at point D, dy/dx = BC/OA.

For the production function

†2.18 $\quad y = f(x),$

the first derivative dy/dx of equation †2.18 is a function that represents the slope, or rate of change in the original production function and is sometimes written as

†2.19 $\quad dy/dx = f'(x)$ or f_1,

where $f'(x)$ or f_1 represents the first derivative of, or the rate of change in the original function. Another way of expressing these relationships is

†2.20 $$dy/dx = f'(x) = f_1 = dTPP/dx = MPP.$$

All expressions refer to the rate of change in the original production or *TPP* function. For the production function

†2.21 $$y = 2x$$

†2.22 $$dy/dx = dTPP/dx = MPP = 2$$

Throughout the domain of this production function, the rate of change is a constant 2. Each additional unit of x produces 2 additional units of y. The first derivative of this production function [$f'(x)$ or f_1] is 2 for all values of x. Note that in this case dy/dx is exactly the same as $\Delta y/\Delta x$. This is because the slope of the function is a constant 2, not dependent on the value of x.

Suppose the production function

†2.23 $$y = bx,$$

where b is any positive number. Again b is the *MPP* of x. The derivative of the production function dy/dx is b. Each incremental unit of x will produce b units of y. If x is increased by 1 unit from any initial level, *TPP* will increase by b units. If b were negative, then *TPP* would decrease, but this would be a silly production function because positive amounts of x would result in negative amounts of y. It is not entirely clear what a negative bushel of corn would look like. Again, b is constant, and dy/dx will always equal $\Delta y/\Delta x$.

Now suppose that the production function is represented by the equation

†2.24 $$y = 50 + 5.93\, x^{0.5}.$$

The *MPP* of x for this function is not the same for every value of x. To calculate the *MPP* at a particular value for x, not only the derivative of the production function is needed, but also how much x is applied. Two simple rules can be used to find the derivative of any production function similar to the one above.

The first rule states that the derivative of any constant value in a function is 0. In this case, the derivative of 50 is 0. The constant is an intercept term that places the function at $x = 0$ on the y axis at 50. A constant does not affect the slope of the function. The second rule is that the derivative of any function of the general form

†2.25 $$y = bx^n$$

can be found by the rule

†2.26 $$dy/dx = nbx^{n-1}$$

where n and b are any numbers. For example, the derivative of the function $y = x^2$ is $dy/dx = 2x$; the derivative of the function $y = 3x^4$ is $dy/dx = 3 \cdot 4 \cdot x^3$ or $12x^3$. If these functions were production functions, their corresponding derivatives would be the corresponding marginal product functions, representing the slopes or rates of change in the original production functions. The derivative for the production function representing corn yield response to nitrogen fertilizer [equation †2.26] is $dy/dx = 0 + 0.5 \cdot 5.93 \cdot x^{-0.5}$, or dy/dx equals $2.965x^{-0.5}$.

A number raised to a negative power is 1 over the number raised to the corresponding positive power: for example,

$$x^{-2} = 1/x^2 \qquad (2.27)$$

In this case

$$dy/dx = 2.965/x^{.5} \qquad (2.28)$$

or

$$dy/dx = 2.965/\sqrt{x} \qquad (2.29)$$

If the amount of x to be applied is known, the corresponding *TPP* is $50 + 5.93x^{0.5}$, and the corresponding *MPP* is $2.965/x^{0.5}$. In this case, *MPP* is specifically linked to the amount of x that is used, as x appears in the first derivative. If this is the case, dy/dx will provide the exact *MPP* but will not be the same as the approximation calculated by $\Delta y/\Delta x$.

Table 2.4 presents *MPP*'s calculated by two methods from yield data obtained from this production function [equation (2.24)]. The first method computes the rate of change in the yields for 40-pound fertilizer increments as was done in the earlier example (Table 2.3). The second method inserts values for nitrogen application levels into the *MPP* function obtained by taking the derivative of the original production function. The values chosen are at the midpoints (20, 60, 100, 140 and 180 pounds of nitrogen per acre).

As is evident from Table 2.4, the results using the two methods are not the same. Method 1 provides the approximate *MPP* at the midpoint. However, for certain fertilizer application levels (for example at 20 pounds per acre) the *MPP* using this first method is very different from the *MPP* obtained by inserting the actual midpoint value into the *MPP* function. This is because the production function is curvilinear, and the slope calculated using method 1 is only a crude approximation of the exact slope of the production function over each 40-pound increment of fertilizer use.

Table 2.4 *MPP* of Nitrogen in the Production of Corn Under Two Alternative Approaches

Quantity of Nitrogen (lb/acre)	Corn Yield (y or *TPP*) (bu/acre)	Average *MPP* Method 1	Exact *MPP* Method 2
0	50.0		
		0.9375	0.6630 (N = 20 lb/Acre)
40	87.5		
		0.3875	0.3827 (N = 60 lb/Acre)
80	103.0		
		0.3000	0.2965 (N = 100 lb/Acre)
120	115.0		
		0.2500	0.2506 (N = 140 lb/Acre)
160	125.0		
		0.2225	0.2212 (N = 180 lb/Acre)
200	133.9		

As is evident from Table 2.4, the results using the two methods are not the same. Method 1 provides the approximate *MPP* at the midpoint. However, for certain fertilizer application levels (for example at 20 pounds per acre) the *MPP* using this first method is very different from the *MPP* obtained by inserting the actual midpoint value into the *MPP* function. This is because the production function is curvilinear, and the slope calculated using method 1 is only a crude approximation of the exact slope of the production function over each 40-pound increment of fertilizer use.

The derivative of the function will provide the exact slope of the function at any selected nitrogen application level. Therefore, the calculated *MPP* values from method 2 are highly accurate for the assumed levels of nitrogen use. Using method 2, the *MPP* can be calculated at any selected level of fertilizer use (including the application levels of 40, 80, 140, 160, and 200 pounds per acre). Basic differential calculus is a powerful tool in agricultural production economics.

Finally, assume that the production function describing corn yield response to nitrogen fertilizer is the one used as the basis for the data contained in Table 2.5. That function was

†2.30 $$y = 0.75x + 0.0042x^2 - 0.000023x^3$$

Following the rules for differentiation, the marginal product function corresponding to equation †2.30 is

†2.31 $$dy/dx = 0.75 + 0.0084x - 0.000069x^2$$

Since *APP* is y/x, the corresponding *APP* function is

†2.32 $$\begin{aligned} y/x &= (0.75x + 0.0042x^2 - 0.000023x^3)/x \\ &= 0.75 + 0.0042x - 0.000023x^2 \end{aligned}$$

Table 2.5 illustrates the exact *APP* and *MPP* values for equation †2.30 obtained by inserting the amount of x (nitrogen) appearing in the first column of the Table into the *MPP* [equation †2.31] and *APP* †equation †2.32].

Table 2.5 Corn Yields, *APP* and *MPP* for $y = 0.75x + 0.0042x^2 - 0.000023x^3$

x (Nitrogen)	y (Corn) or *TPP*	*APP* of x, y/x	*MPP* of x, dy/dx
0	0.0	undefined	0.7500
20	16.496	0.8248	0.8904
40	35.248	0.8812	0.9756
60	55.152	0.9192	1.0056
80	75.104	0.9388	0.9804
100	94.000	0.9400	0.9000
120	110.736	0.9228	0.7644
140	124.208	0.8872	0.5736
160	133.312	0.8332	0.3276
180	136.944	0.7608	0.0264
200	134.000	0.6700	−0.3300
220	123.376	0.5608	−0.7416
240	103.968	0.4332	−1.2084

2.6 A Neoclassical Production Function

Figure 2.3 illustrates a neoclassical production function that has long been popular for describing production relationships in agriculture. With this production function, as the

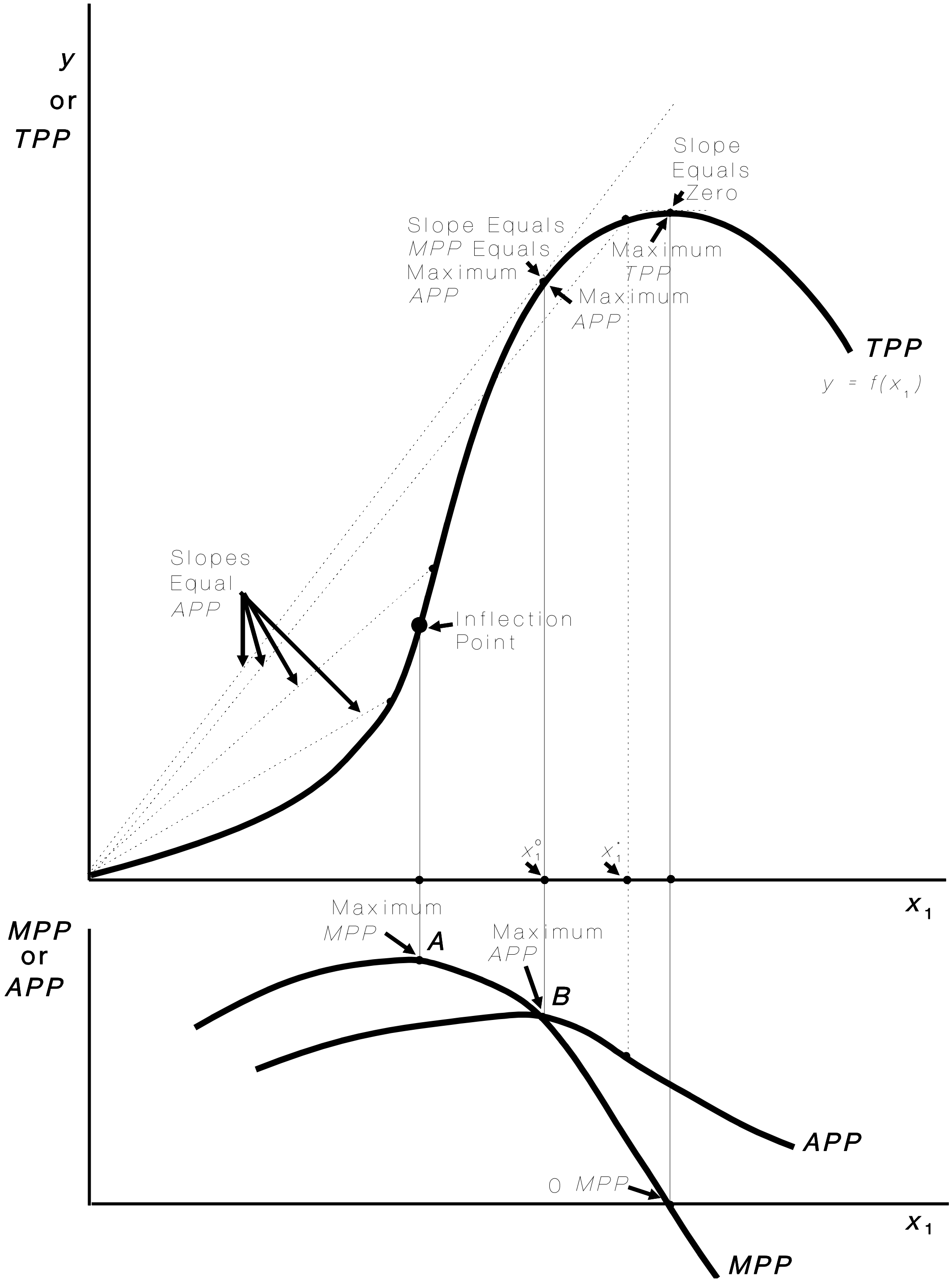

Figure 2.3 A Neoclassical Production Function

use of input x_1 increases, the productivity of the input at first also increases. The function turns upward, or increases, at first at an increasing rate. Then a point called the *inflection point* occurs. This is where the function changes from increasing at an increasing rate to increasing at a decreasing rate. Another way of saying this is that the function is convex to the horizontal axis prior to the inflection point, but concave to the horizontal axis after the inflection point. The inflection point marks the end of increasing marginal returns and the start of diminishing marginal returns. Finally, the function reaches a maximum and begins to turn downward. Beyond the maximum, increases in the use of the variable input x_1 result in a decrease in total output (*TPP*). This would occur in an instance where a farmer applied so much fertilizer that it was actually detrimental to crop yields.

2.7 *MPP* and *APP* for the Neoclassical Function

The *MPP* function changes as the use of input x_1 increases. At first, as the productivity of input x_1 increases, so does its marginal product, and the corresponding *MPP* function must be increasing (Figure 2.3). The inflection point marks the maximum marginal product. It is here that the productivity of the incremental unit of the input x_1 is at its greatest. After the inflection point, the marginal product of x_1 declines and the *MPP* function must also be decreasing. The marginal product of x_1 is zero at the point of output maximization, and negative at higher levels. Therefore, the *MPP* function is zero at the point of output maximization, and negative thereafter.

Average physical product (*APP*) also changes as the use of x_1 increases, although *APP* is never negative. As indicated earlier, *APP* is the ratio of output to input, in this case y/x_1 or TPP/x_1. Since this is the case, *APP* for a selected point on the production function can be illustrated by drawing a line (ray) out of the origin of the graph to the selected point. The slope of this line is y/x_1 and corresponds to the values of y and x_1 for the production function. If the point selected on the function is for some value for x_1 called x_1^*, then the *APP* at x_1^* is y/x_1^*.

APP reaches a maximum at a point after the inflection point but before the point in which output is maximized. Figure 2.3 illustrates several lines drawn out of the origin. The line with the greatest slope is tangent to the production function at that point. Therefore it also represents the slope of the production function at that point. The slope of each line drawn from the origin to a point on the production function represents the *APP* for the function at that point, but only one line is tangent to and thus also represents the slope of the production function at that point. It is here where marginal product must equal average product, *APP* must equal *MPP*, and $y/x = dy/dx$.

Call the point $x_1°$ where $y/x = dy/dx$. At any point less than $x_1°$, the slope of the production function is greater than the slope of the line drawn from the origin to the point. Hence *APP* must be less than *MPP* prior to $x_1°$. As the use of x_1 increases toward $x_1°$, *APP* increases, as does the slope of the line drawn from the origin. After $x_1°$, the slope of the production function is less than the slope of the line drawn from the origin to the point. Hence *MPP* must be less than *APP* after $x_1°$. As the use of x_1 increases beyond $x_1°$, the slope of the line drawn from the origin to the point declines, and *APP* must decline beyond $x_1°$. The slope of that line never becomes negative, and *APP* never becomes negative.

However, a line drawn tangent to the production function represents *MPP* and will have a negative slope beyond the point of output maximization. *APP* is always non-negative, but *MPP* is negative beyond the point of output maximization.

Figure 2.3 also illustrates the relationships that exist between the *APP* and the *MPP* function for the neoclassical production function. The *MPP* function first increases as the use of the input is increased, until the inflection point of the underlying production function is reached (point A). Here the *MPP* function reaches its maximum. After this point, *MPP* declines, reaches zero when output is maximum (point C), and then turns negative. The *APP* function increases past the inflection point of the underlying production function until it reaches the *MPP* function (point B). After point B, *APP* declines, but never becomes negative.

The relationships that hold between *APP* and *MPP* can be proven using the composite function rule for differentiation. Notice that

†2.33 $y = (y/x)\cdot x$, or $TPP = APP \cdot x$ in the original production or *TPP* function.

†2.34 $dy/dx = y/x + [d(y/x)/dx]\cdot x$

or, equivalently, $MPP = APP + (\text{slope of } APP)x$.

If *APP* is increasing and therefore has a positive slope, then *MPP* must be greater than *APP*. If *APP* is decreasing and therefore has a negative slope, *MPP* must be less than *APP*. If *APP* has a zero slope, such as would be the case where it is maximum, *MPP* and *APP* must be equal.

Figure 2.4 illustrates the *TPP*, *MPP*, and *APP* curves that are generated from the data contained in Table 2.5. The maximum of the production function corresponds to an output level of 136.96 bushels of corn per acre, using a nitrogen application rate (x) of 181.60 pounds per acre. The inflection point of this production function corresponding with the maximum *MPP* occurs at an output level of 56.03 bushels of corn (y), with a corresponding nitrogen application rate of 60.86 pounds per acre, The *APP* maximum, where *MPP* intersects *APP*, occurs at an output level of 85.98 bushels of corn per acre, with a corresponding nitrogen (x) application rate of 91.30 bushels per acre. The actual production function illustrated from the data contained in Table 2.5 appears quite similar to the neoclassical function illustrated in Figure 2.3.

2.8 Sign, Slope and Curvature

By repeatedly differentiating a production function, it is possible to determine accurately the shape of the corresponding *MPP* function. For the production function

†2.35 $y = f(x)$

the first derivative represents the corresponding *MPP* function

†2.36 $dy/dx = f'(x) = f_1 = MPP$

Insert a value for x into the function $f'(x)$ [equation †2.36]. If $f'(x)$ (or dy/dx or *MPP*) is positive, then incremental units of input produce additional output. Since *MPP* is negative after the production function reaches its maximum, a positive sign on $f'(x)$ indicates that the underlying production function has a positive slope and has not yet achieved a maximum. If $f'(x)$ is negative, the production function is downsloping, having already achieved its maximum. The sign on the first derivative of the production function indicates if the slope of the production function is positive or negative and if *MPP* lies above or below the horizontal axis. If *MPP* is zero, then $f'(x)$ is also zero, and the production function is likely either constant or at its maximum. Figure 2.5 illustrates seven instances where the first derivative of the *TPP* function is positive [(a) to (g)] and seven instances where the first derivative is negative [(h) to (n)].

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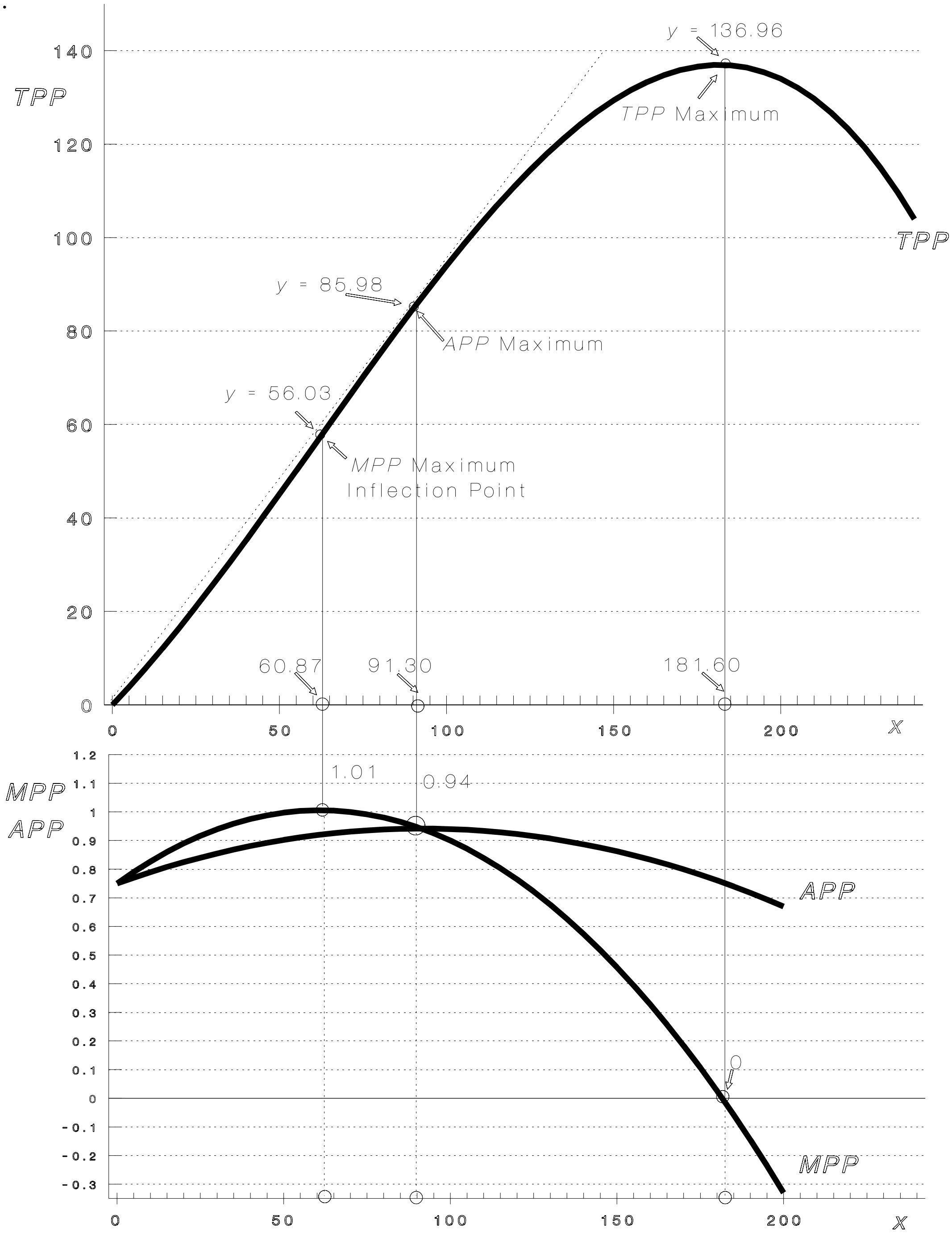

Figure 2.4 ***TPP*, *MPP*, and *APP* For Corn (*y*) Response to Nitrogen (*x*) Based on Data Contained in Table 2.5**

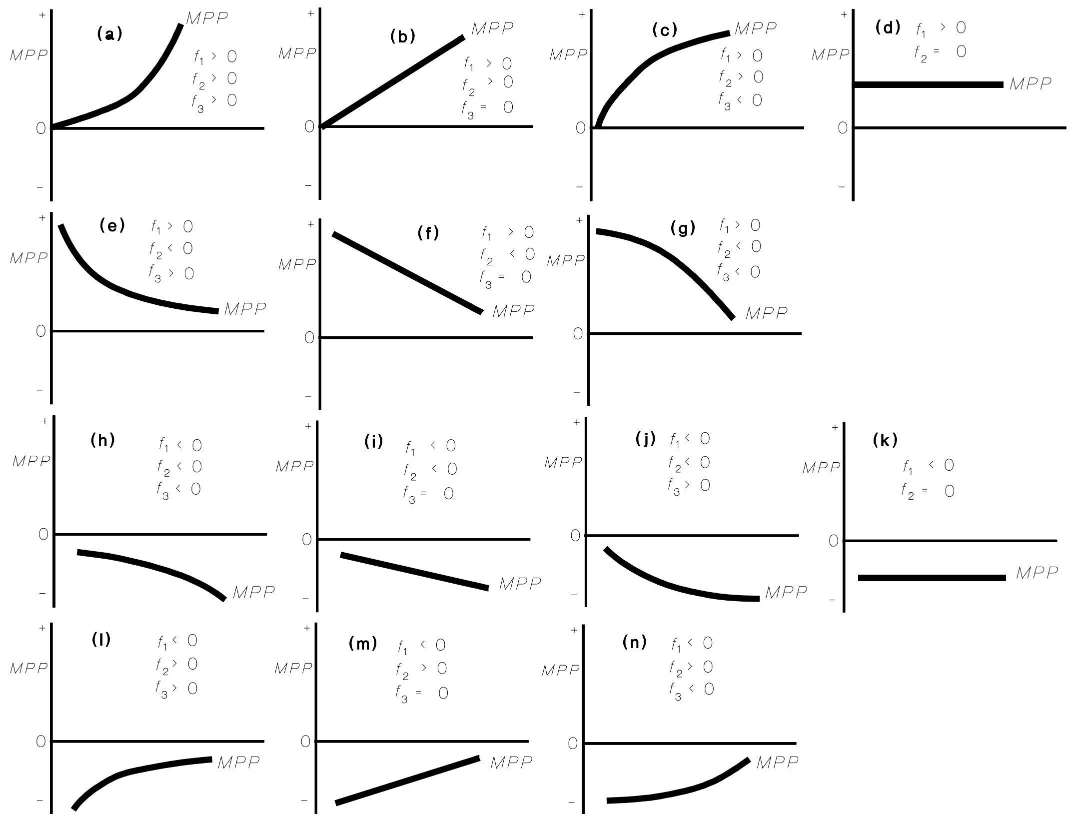

Figure 2.5 ***MPP*'s for the Production Function *y = f(x)***

$f_1 = MPP$; f_2 = slope of MPP; f_3 = curvature of MPP

The first derivative of the *TPP* function could also be zero at the point where the *TPP* function is minimum. The sign on the second derivative of the *TPP* function is used to determine if the *TPP* function is at a maximum or a minimum. If the first derivative of the *TPP* function is zero and the second derivative is negative, the production function is at its maximum. If the first derivative of the *TPP* function is zero, and the second derivative is positive, the production function is at its minimum point. If both the first and second derivatives are zero, the function is at an inflection point, or changing from convex to the horizontal axis to concave to the horizontal axis. However, all inflection points do not necessarily have first derivatives of zero. Finally, if the first derivative is zero and the second derivative does not exist, the production function is constant.

The second derivative of the production function is the first derivative of the *MPP* function, or slope of the *MPP* function. The second derivative (d^2y/dx^2 or $f0(x)$ or f_2) is obtained by again differentiating the production function.

†2.37 $$d^2y/dx^2 = f0(x) = f_2 = dMPP/dx$$

If equation †2.37 is positive for a particular value of x, then *MPP* is increasing at that particular point. A negative sign indicates that *MPP* is decreasing at that particular point. If $f''(x)$ is zero, *MPP* is likely at a maximum at that point. In figure 2.4, the first derivative of the *MPP* function (second derivative of the *TPP* function) is positive in (a), (b), and (c), (l), (m),and (n); negative in (e), (f), (g), (h), (i), and (j), and zero in (d) and (k).

The second derivative of the *MPP* function represents the curvature of *MPP* and is the third derivative of the original production (or *TPP*) function. It is obtained by again differentiating the original production function

†2.38 $d^3y/dx^3 = f'''(x) = f_3 = d^2MPP/dx^2$

The sign on $f'''(x)$ for a particular value of x indicates the rate of change in *MPP* at that particular point. If *MPP* is in the postive quadrant and $f'''(x)$ is positive, *MPP* is increasing at an increasing rate [(a) in Figure 2.5] or decreasing at a decreasing rate (e). If *MPP* is in the negative quadrant, a positive $f'''(x)$ indicates that MPP is either decreasing at a decreasing rate (j) or increasing at a decreasing rate (l).

When *MPP* is in the positive quadrant, a negative sign on $f'''(x)$ indicates that *MPP* is either increasing at a decreasing rate (c), or decreasing at an increasing rate (g). When *MPP* is in the negative quadrant, a negative sign on $f'''(x)$ indicates that *MPP* is decreasing at an increasing rate (h) or increasing at an increasing rate (n).

If $f'''(x)$ is zero, *MPP* has a constant slope with no curvature as is the case in (f), (l), and (m). If *MPP* is constant, $f'''(x)$ does not exist.

A similar approach might be used for *APP*. *APP* equals y/x, and if y and x are positive, then *APP* must also be positive. As indicated earlier, the slope of *APP* is

†2.39 $d(y/x)/dx = f'(y/x) = dAPP/dx$

For a particular value of x, a positive sign indicates a positive slope and a negative sign a negative slope.

The curvature of *APP* can be represented by

†2.40 $d^2(y/x)/dx^2 = f''(y/x) = d^2APP/dx^2$

For a particular value of x, a positive sign indicates that *APP* is increasing at an increasing rate, or decreasing at a decreasing rate. A negative sign on equation †2.40 indicates that *APP* is increasing at a decreasing rate, or decreasing at an increasing rate. A zero indicates an *APP* of constant slope. The third derivative of *APP* would represent the rate of change in the curvature of *APP*.

Here are some examples of how these rules can be applied to a specific production function representing corn yield response to nitrogen fertilizer. Suppose the production function

†2.41 $y = 50 + 5.93\,x^{0.5}$

where

y = corn yield in bushels per acre
x = pounds of nitrogen applied per acre

†2.42 $MPP = f'(x) = 2.965\,x^{-0.5} > 0$

For equation †2.41 , *MPP* is always positive for any positive level of input use, as indicated by the sign on equation †2.42 . If additional nitrogen is applied, some additional response in terms of increased yield will always result. If x is positive, *MPP* is positive and the production function has not reached a maximum.

†2.43 $dMPP/dx = f\text{O}(x) = \,!\,1.48\, x^{!1.5} < 0$

If equation †2.43 is negative, *MPP* is slopes downward. Each additional pound of nitrogen that is applied will produce less and less additional corn yield. Thus the law of diminishing (MARGINAL) returns holds for this production function throughout its range.

†2.44 $d^2MPP/dx^2 = f\text{“}(x) = 2.22\, x^{!2.5} > 0$

If equation †2.44 holds, the *MPP* function is decreasing at a decreasing rate, coming closer and closer to the horizontal axis but never reaching or intersecting it. This is not surprising, given that incremental pounds of nitrogen always produce a positive response in terms of additional corn.

†2.45 $APP = y/x = 50/x + 5.93x^{!0.5}$

$= 50\, x^{!1} + 5.93x^{!0.5} > 0$

If x is positive, *APP* is positive. Corn produced per pound of nitrogen fertilizer is always positive [equation †2.45].

†2.46 $dAPP/dx = \mathrm{d}(y/x)/dx = \,!\,50\, x^{!2} \,!\, 2.97\, x^{!1.5} < 0$

If x is positive, *APP* is sloped downward. As the use of nitrogen increases, the average product per unit of nitrogen declines [Equation †2.46].

†2.47 $d^2APP/dx^2 = d^2(y/x)/dx^2 = 100x^{!3} + 4.45\, x^{!2.5} > 0$

If x is positive, *APP* is also decreasing at a decreasing rate. As the use of nitrogen increases, the average product per unit of nitrogen decreases but at a decreasing rate [equation †2.47].

2.9 A Single-Input Production Elasticity

The term *elasticity* is used by economists when discussing relationships between two variables. An elasticity is a number that represents the ratio of two percentages. Any elasticity is a pure number in that it has no units.

The elasticity of production is defined as the percentage change in output divided by the percentage change in input, as the level of input use is changed. Suppose that xN represents some original level of input use that produces yN units of output. The use of x is then increased to some new amount called xO, which in turn produces yO units of output. The elasticity of production (E_p) is defined by the formula

†2.48 $E_p = [(y\text{N} \,!\, y\text{O})/y]/[(x\text{N} \,!\, x\text{O})/x].$

where y, yO, x, and xO are as defined previously, and x and y represent mid values between the old and new levels of inputs and outputs. Thus

†2.49 $x = (x\text{N} + x\text{O})/2$

and $y = (y' + y'')/2$

Since the elasticity of production is the ratio of two percentages, it does not depend on the specific units in which the input and output are measured. For example, suppose that y represents corn yield in bushels per acre, and x represents nitrogen in pounds per acre. Then suppose that corn yield is instead measured in terms of liters per hectare, and nitrogen was measured in terms of kilograms per hectare. If the same amount of nitrogen is applied in both instances, the calculated value for the elasticity of production will be the same, regardless of the units in which y and x are measured.

Another way of expressing the elasticity of production is

†2.50 $E_p = (\Delta y/y)/(\Delta x/x)$

where $\Delta y = y' - y''$

and $\Delta x = x' - x''$

The elasticity of production is one way of measuring how responsive the production function is to changes in the use of the input. A large elasticity (for example, an elasticity of production greater than 1) implies that the output responds strongly to increases in the use of the input. An elasticity of production of between zero and 1 suggests that output will increase as a result of the use of x, but the smaller the elasticity, the less the response in terms of increased output. A negative elasticity of production implies that as the level of input use increases, output will actually decline, not increase.

The elasticity of production can also be defined in terms of the relationship between *MPP* and *APP*. The following relationships hold. First

†2.51 $E_p = (\Delta y/y)/(\Delta x/x)$

Equation †2.51 might also be written as

†2.52 $E_p = (\Delta y/\Delta x)\cdot(x/y)$

Notice that

†2.53 $\Delta y/\Delta x = MPP$

and that

†2.54 $x/y = 1/APP$

Thus

†2.55 $E_p = MPP/APP$

Notice that a large elasticity of production indicates that *MPP* is very large relative to *APP*. In other words, output occurring from the last incremental unit of fertilizer is very great relative to the average output obtained from all units of fertilizer. If the elasticity of production is very small, output from the last incremental unit of fertilizer is small relative to the average productivity of all units of fertilizer.

2.10 Elasticities of Production for a Neoclassical Production Function

A unique series of elasticities of production exist for the neoclassical production function, as a result of the relationships that exist between *MPP* and *APP*. These are illustrated in Figure 2.6 and can be summarized as follows

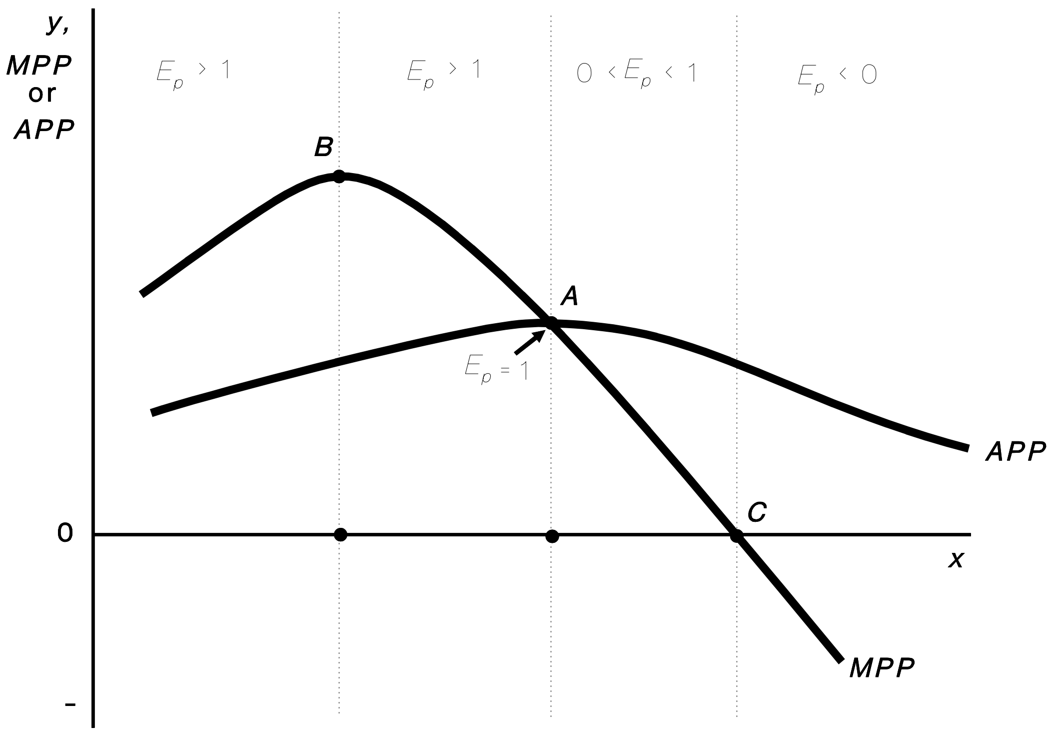

Figure 2.6 ***MPP, APP*** **and the Elasticity of Production**

1. The elasticity of production is greater than 1 until the point is reached where *MPP* = *APP* (point *A*).

2. The elasticity of production is greatest when the ratio of *MPP* to *APP* is greatest. For the neoclassical production function, this normally occurs when *MPP* reaches its maximum at the inflection point of the production function (point *B*).

3. The elasticity of production is less than 1 beyond the point where *MPP* = *APP* (point *A*).

4. The elasticity of production is zero when *MPP* is zero. Note that *APP* must always be positive (point *C*).

5. The elasticity of production is negative when *MPP* is negative and, of course, output is declining (beyond point *C*). If the production function is decreasing, *MPP* and the elasticity of production are negative. Again, *APP* must always be positive.

6. A unique characteristic of the neoclassical production function is that as the level of input use is increased, the relationship between *MPP* and *APP* is continually changing, and therefore the ratio of *MPP* to *APP* must also vary. Since $E_p = MPP/APP$, the elasticity of production too must vary continually as the use of the input increases. This is a characteristic of the neoclassical production function, which in general is not true for some other production functions.

2.11 Further Topics on the Elasticity of Production.

The expression $\Delta y/\Delta x$ is only an approximation of the true *MPP* of the production function for a specific amount of the input *x*. The actual *MPP* at a specific point is better represented by inserting the value of *x* into the marginal product function dy/dx.

The elasticity of production for a specific level of *x* might be obtained by determining the value for dy/dx for that level of *x* and then obtaining the elasticity of production from the expression

†2.56 $\qquad E_p = (dy/dx) \cdot x/y$

Now suppose that instead of the neoclassical production function, a simple linear relationship exists between *y* and *x*. Thus

†2.57 $\qquad TPP = y = bx$

where *b* is some positive number. Then $dy/dx = b$, but note also that since $y = bx$, then $y/x = bx/x = b$. Thus *MPP* (dy/dx) = *APP* $(y/x) = b$. Hence, $MPP/APP = b/b = 1$.

The elasticity of production for any such function is 1. This means that a given percentage increase in the use of the input *x* will result in exactly the same percentage increase in the output *y*. Moreover, any production function in which the returns to the variable input are equal to some constant number will have an elasticity of production equal to 1.

Now suppose a slightly different production function

†2.58 $\qquad y = a\sqrt{x}$

Another way of writing equtation †2.58 is

†2.59 $\qquad y = ax^{0.5}$

In this case

†2.60 $\qquad dy/dx = 0.5\, ax^{-0.5}$

And

†2.61 $\qquad y/x = ax^{-0.5}$

Thus, $(dy/dx)/(y/x) = 0.5$

Hence the elasticity of production is 0.5. This means that for any level of input use *MPP* will be precisely one half of *APP*. In general, the elasticity of production will be *b* for any production function of the form

†2.62 $\qquad y = ax^b$

where *a* and *b* are any numbers. Notice that

†2.63 $\qquad dy/dx = bax^{b-1}$

and that

†2.64 $\qquad y/x = ax^b/x = ax^b x^{-1} = ax^{b-1}$.

(Another way of writing the expression $1/x$ is x^{-1}. Therefore, $y/x = yx^{-1}$. But $y = ax^b$, and, as a result, $x^b x^{-1} = x^{b-1}$.)

Thus the ratio of *MPP* to *APP* — the elasticity of production — for such a function is always equal to the constant b. This is not the same as the relationship that exists between *MPP* and *APP* for the neoclassical production function in which the ratio is not constant but continually changing as the use of x increases.

2.12 Concluding Comments

This chapter has outlined in considerable detail the physical or technical relationships underlying the factor-product model. A production function was developed using tabular, graphical, and mathematical tools, with illustrations from agriculture. The law of diminishing *MARGINAL* returns was introduced. Marginal and average physical product concepts were developed. The rules of calculus for determining if a function is at a maximum or minimum were outlined, using a total physical product and marginal physical product concepts to illustrate the application. Finally, the concept of an elasticity of production was introduced, and the elasticity of production was linked to the marginal and average product functions.

Problems and Exercises

1. Suppose the following production function data. Fill in the blanks.

x (Input)	y (Output)	*MPP*	*APP*
0	0		___
		___	
10	50		___
		___	
25	75		___
		___	
40	80		___
		___	
50	85		___

2. For the following production functions, does the law of diminishing returns hold?

    a. $y = x^{0.2}$
    b. $y = 3x$
    c. $y = x^3$
    d. $y = 6x - 0.10x^2$

3. Find the corresponding *MPP* and *APP* functions for the production functions given in problem number 2.

4. Assume a general multiplicative production function of the form
$y = 2x^b$

Derive the corresponding *MPP* and *APP* functions, and draw on a sheet of graph paper *TPP*, *APP* and *MPP* when the value of b is

a. 5
b. 3
c. 2
d. 1.5
e. 1.0
f. 0.7
g. 0.3
h. 0
i. -0.5
j. -1.0

Be sure to show the sign, slope and curvature of *MPP* and *APP*. What is the value for the elasticity of production in each case? Notice that the curves remain at fixed proportion from each other.

5. Graph the production function

$$y = 0.4x + 0.09x^2 - 0.003x^3$$

for values of x between 0 and 20. Derive and graph the corresponding *MPP* and *APP*. What is the algebraic expression for the elasticity of production in this case? Is the elasticity of production constant or variable for this function? Explain.

6. Suppose that the coefficients or parameters of a production function of the polynomial form are to be found. The production function is

$$y = ax + bx^2 + cx^3$$

where y = corn yield in bushels per acre

x = nitrogen application in pounds per acre

a, b and c are coefficients or unknown parameters

The production function should produce a corn yield of 150 bushels per acre when 200 pounds of nitrogen is applied to an acre. This should be the maximum corn yield (*MPP* = 0). The maximum *APP* should occur at a nitrogen application rate of 125 pounds per acre. Find the parameters a, b and c for a production function meeting these restrictions. *Hint:* First find the equation for *APP* and *MPP*, and the equations representing maximum *APP* and zero *MPP*. Then insert the correct nitrogen application levels in the three equations representing *TPP*, maximum *APP* and zero *MPP*. There are three equations in three unknowns (a, b, and c). Solve this system for a, b, and c.